ESD-TR-66-66 MTR-32

NUMERICAL COMPARISON OF EXACT, UNIFORM, AND NON-UNIFORM ASYMPTOTIC SOLUTION OF THE REDUCED WAVE EQUATION NEAR A CAUSTIC

JUNE 1966

R. M. Lewis (Consultant)

L. J. Kaplan

Prepared for

SPACE DEFENSE SYSTEM PROGRAM OFFICE (496L/474L)

ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
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L. G. Hanscom Field, Bedford, Massachusetts

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Project 4966
Prepared by
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FOREWORD

We wish to thank D. Ludwig and J. D. R. Kramer for their interesting comments and suggestions. We also wish to thank Mrs. M. J. Carey for her able programming.

REVIEW AND APPROVAL

This technical report has been reviewed and is approved.

Harry R. Betz

HARRY R. BETZ, 2/Lt., USAF
Project Manager
496L/474L System Program Office

ABSTRACT

A numerical comparison of the exact solution, the uniform asymptotic solution, and the non-uniform asymptotic solution of a field that produces a circular caustic is performed. It is seen that the uniform asymptotic solution is accurate even at the caustic for moderate values of ka. Curves and data are given as a function of ka and distances from the caustic. The asymptotic expressions are derived both from the exact solution and from the physics of the circular caustic.

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DERIVATION

The uniform asymptotic expansion developed in Lewis' paper* is valid in the neighborhood of a caustic, and it reduces to the usual geometrical optics representation away from the caustic. In Appendix I, these expansions are calculated for the circular caustic. Herein, an exact solution of

$$\left(\nabla^2 + k^2\right) \mu = 0 , \tag{1}$$

whose asymptotic expansions agree with that of Appendix I, enables us to obtain a numerical comparison of an exact solution and its uniform and non-uniform approximations.

An exact solution of (1) in two dimensions, where polar coordinates are used, is the function

$$\mu = e^{ika\phi} J_{ka}(kr). \tag{2}$$

Here $J_n(z)$ is the Bessel function of order n. We obtain a uniform asymptotic expansion of (2) by using the following formula, given in Asymptotic Approximations by H. Jeffreys:†

$$J_n\left(n \operatorname{sech} \mu\right) \sim \left(\frac{4\xi}{\tan^2 \mu}\right)^{1/4} n^{-1/3} \operatorname{Ai}\left(n^{2/3} \xi\right) ; \ \xi = \frac{3}{2} \left(\mu - \tanh \mu\right). \tag{3}$$

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* R. M. Lewis, Uniform Representation of Geometrical Optics Fields Near a Caustic, The MITRE Corporation, ESD-TR-65-404, Bedford, Mass., May 1966.

† H. Jeffreys, Asymptotic Approximations, England, Oxford University Press, 1962, p. 84, Equation (36).

In (3), Ai is the Airy function defined and discussed in Jeffreys' work* and in Appendix II. The formula is valid for large n.

In order to apply (3) to (2), one sets

$$z - \operatorname{sech}\mu = (\cos v)^{-1}, \quad v = 1\mu \tag{4}$$

Thus,

$$2/3\,\zeta^{3/2} = -iv - \tanh(-iv) = i(\tan v - v) = i\left(\sqrt{z^2 - 1} - \cos^{-1} 1/z\right),$$

$$\tanh^2 \mu = 1 - \operatorname{sech}^2 \mu = 1 - z^2, \tag{5}$$

and

$$J_n^{\prime}(nz) \sim \left[\frac{4\zeta}{1 - z^2}\right]^{1/4} n^{-1/3}\, \mathrm{Ai}\left(n^{2/3}\zeta\right). \tag{6}$$

Here

$$-\zeta = 3/2\left[\sqrt{z^2 - 1} - \cos^{-1}(1/z)\right]^{2/3} \tag{7}$$

in agreement with (5). Let $n = ka$, $z = r/a$ and apply (6) to (2). The result is

$$\mu \sim e^{ika\phi}\left[\frac{4\zeta}{1 - r^2/a^2}\right]^{1/4} (ka)^{-1/3}\, \mathrm{Ai}\left[(ka)^{2/3}\zeta\right]; \tag{8}$$

or

$$\mu \sim e^{ik\theta} g\, \mathrm{Ai}\left(-k^{2/3}\rho\right). \tag{9}$$

*See page 1, second reference.

Here

$$\theta = a\phi \ , \tag{10}$$

$$\rho = -a^{2/3} = (3/2\,\Psi)^{2/3}\ ,\ 2/3\,\rho^{3/2} = \Psi = \sqrt{r^2 - a^2} - a\cos^{-1}(a/r)\ , \tag{11}$$

and

$$g = (ka)^{-1/3}\left[\frac{-4\zeta}{r^2/a^2 - 1}\right]^{1/4} = \left[\frac{4k^{2/3}\rho}{k^2\left(r^2 - a^2\right)}\right]^{1/4} . \tag{12}$$

The non-uniform expansion is obtained from (9) by using the asymptotic expansion

$$Ai\,(-\zeta) \sim \frac{1}{\sqrt{\pi}}\,\zeta^{-1/4}\sin\left(2/3\,\zeta^{3/2} + \pi/4\right);\ -\pi/3 < \arg\zeta < \pi/3;\ |\zeta| \to \infty \tag{13}$$

of the Airy function.* Let

$\zeta = k^{2/3}\rho$. Then $2/3\,\zeta^{3/2} = k\Psi$ and

$$\mu \sim \frac{1}{\sqrt{\pi}}\,e^{ika\phi}\left[\frac{4}{k^2(r^2 - a^2)}\right]^{1/4} \sin\left(k\Psi + \pi/4\right)\ , \tag{14}$$

or

$$\mu \sim \frac{1}{2\pi i}\left[k^2\left(r^2 - a^2\right)\right]^{-1/4}\left\{e^{ik(a\phi+\Psi)+i\,\pi/4} - e^{ik(a\phi-\Psi)-i\,\pi/4}\right\} . \tag{15}$$

This result may be interpreted in terms of geometrical optics with the aid of Figure 1.

Two rays, each tangent to the circular caustic of radius a, intersect at the point X whose polar coordinates are r, ϕ. The two phases associated with these rays are

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* See page 1, second reference, p. 33, Equation (30).

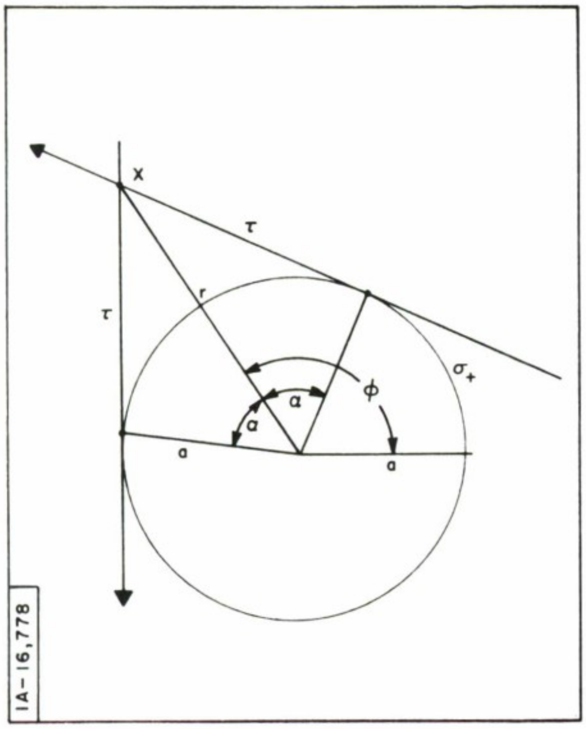

Figure 1. Circular Caustic Corresponding to Equation (15)

$$S_{\pm} = \sigma_{\pm} \pm \tau \, , \tag{16}$$

where

$$\sigma_{\pm} = a(\phi \mp \alpha) = a\phi \mp a \cos^{-1}\left(\frac{a}{r}\right), \tau = \sqrt{r^2 - a^2} \, . \tag{17}$$

We see from (11) that

$$a\phi \pm \Psi = a\phi \mp a \cos^{-1}\left(\frac{a}{r}\right) \pm \tau = \sigma_{\pm} \pm \tau = S_{\pm} \quad , \tag{18}$$

hence, (15) can be written in the form

$$\mu \sim \frac{e^{i\pi/4}}{\sqrt{2\pi k}} \left[z_{-} e^{iks_{-}} + z_{+} e^{iks_{+} - i\pi/2} \right], \tag{19}$$

where

$$z_- = z_+ = \tau^{-1/2} \quad . \tag{20}$$

In (20), the amplitude and the phase of the incoming and outgoing waves are clearly exhibited. The amplitudes become infinite at the caustic ($\tau = 0$), and the phase-shift factor $e^{-i\pi/2}$ multiplies the outgoing wave. The wave fronts are the spirals $S_{\pm}$ = constant.

Here, the uniform expansion (9-12) has been obtained directly from the exact solution (2). In general, of course, the exact solution is not known. Then, it is necessary to use the method developed in Lewis' report* to obtain the uniform expansion. In Appendix I, that method is applied to this problem, and a uniform expansion is obtained that agrees exactly with (9-12).

Numerical computations and graphs of the exact, uniform, and non-uniform results will not be obtained. For this purpose, $\phi = 0$ may be taken, since $e^{ika\phi}$ is a common factor of all three results. It is noted that ka, kr, $k\Psi$, and $k^{2/3}\rho$ are dimensionless, therefore let $k=1$. The exact, uniform, and non-uniform results are denoted by μ_1, μ_2, and μ_3. Then,

$$\mu_1 = J_a(r), \tag{21}$$

$$\mu_2 = \left(\frac{4\rho}{r^2 - a^2}\right)^{1/4} Ai(-\rho) , \tag{22}$$

and

$$\mu_3 = \sqrt{\frac{2}{\pi}} \left(\frac{1}{r^2 - a^2}\right)^{1/4} \sin(\Psi + \pi/4) . \tag{23}$$

---

* See page 1, first reference.

Here

$$\Psi = \sqrt{r^2 - a^2} - a \cos^{-1} (a/r) , \tag{24}$$

and

$$\rho = \left(\frac{3}{2} \Psi\right)^{2/3} . \tag{25}$$

SECTION II

NUMERICAL RESULTS

These three functions were programmed for several values of ka as a function of r/a. Figure 2 shows these curves for ka = 2, 5, and 10. For these values of ka, curves of the exact and uniform solutions are indistinguishable, and only curves of μ_1 and μ_3 are shown. To indicate the errors in the uniform approximations, a deviation curve is plotted for ka = 1 and ka = 2 in Figure 3. Figure 4 is the same as Figure 2 except that ka = 1/2, 1/4, 1/8, and 1/16.

For these values of ka, μ_2 has diverged sufficiently from μ_1 so that three curves can be distinguished. From these curves the following conclusions are clear. The non-uniform approximation is very accurate

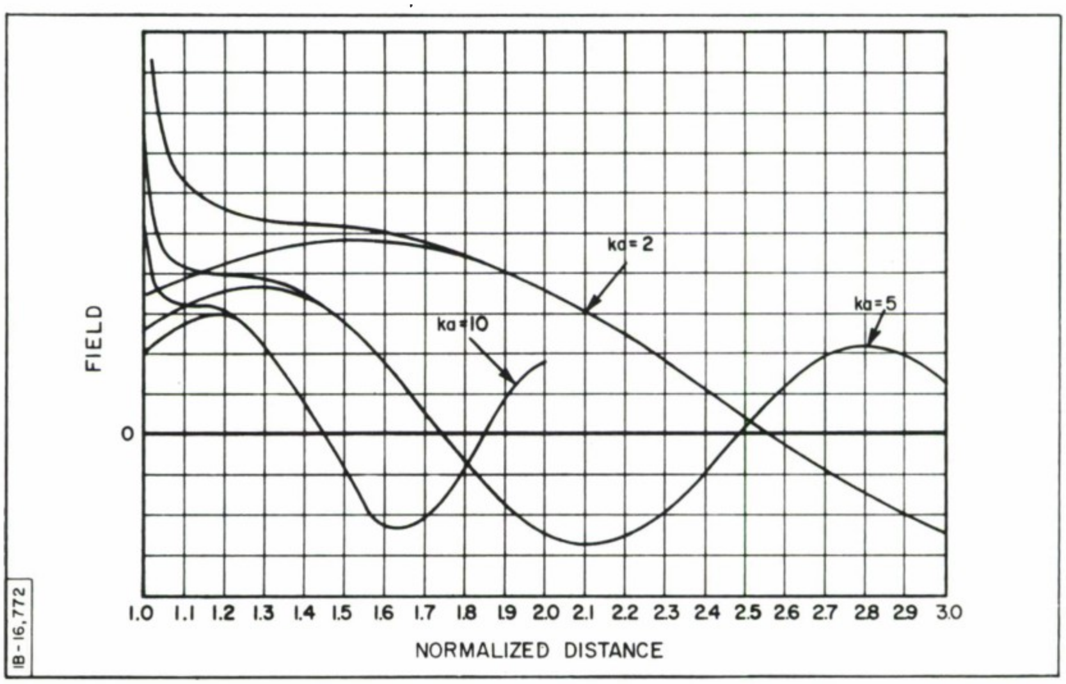

Figure 2. Plot of Field versus Normalized Distance for Various ka

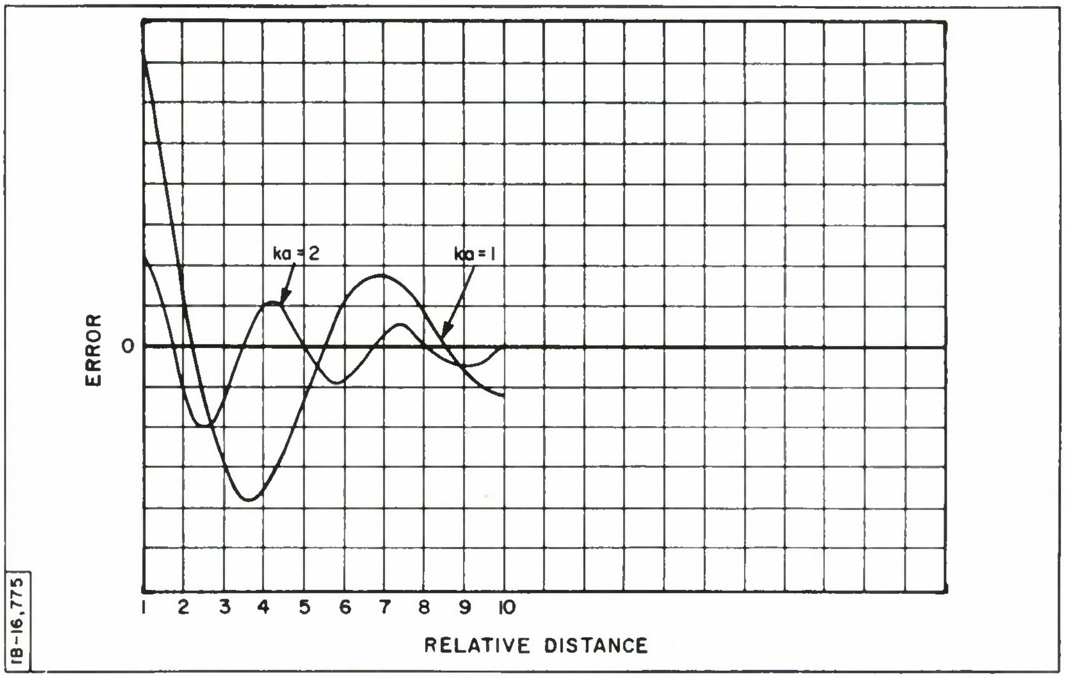

Figure 3. Plot of Error versus Relative Distance for Two Values of ka

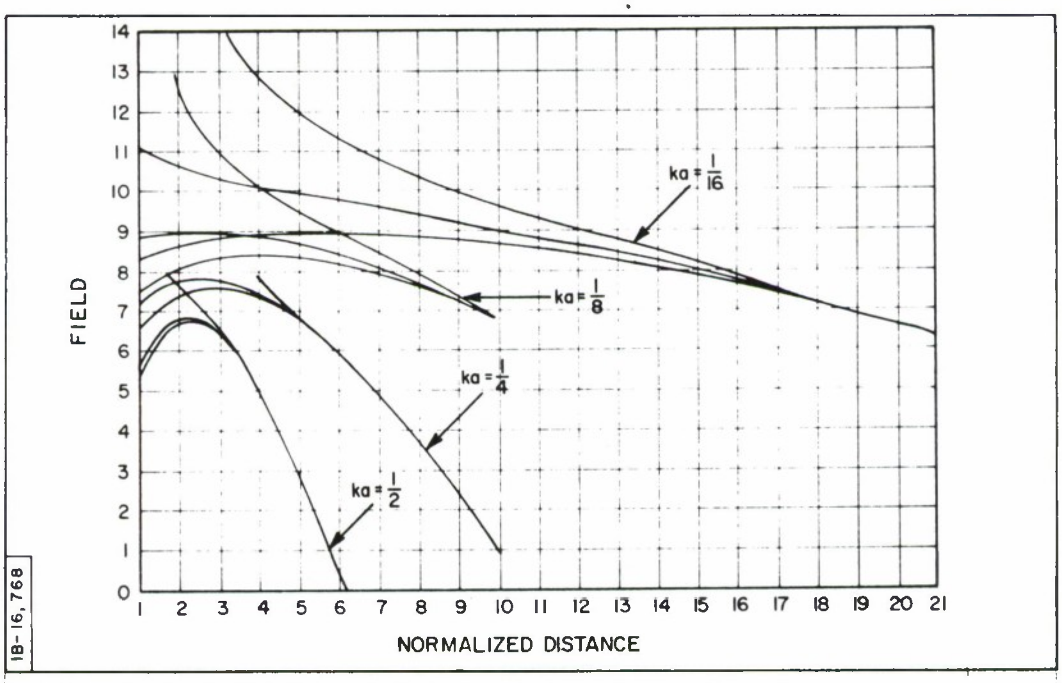

Figure 4. Plot of Field versus Normalized Distance for Various ka

for sufficiently large r/a and all ka. The uniform approximation is good for almost all r/a even for quite small ka. To indicate the actual errors in the uniform approximation, the maximum full-scale percentage error for various ka is indicated in Table I. This is defined as

$$\frac{\text{maximum} \, |\mu_2 - \mu_1| \, 100}{\text{maximum} \ \mu_1} \ .$$

From the curves of Figure 3, it can be seen that the deviation from the exact solution at a given point may be considerably smaller than this value.

Examination of the field on the caustic using (21), (22), and (23) leads to the exact solution $\mu_1 = J_{ka}(ka)$, while the uniform approximation is indeterminate and the non-uniform approximation is singular. In Appendix III, an expression is developed for the uniform approximation (26).

Table I

Percentage Error for Various ka

ka	Percentage Error
1/64	87.5
1/32	57.6
1/16	31.6
1/8	17.3
1/4	8.05
1/2	3.3
1	1.7
2	0.45
3	0.27
5	0.18
10	0.082
20	0.037
50	0.017

$$u = \frac{2^{1/3}}{a^{1/3}} \quad A_i \left[\frac{-2^{1/3}\epsilon}{a^{1/3}} \right] \tag{26}$$

In Figure 5, the actual percentage error at the caustic as a function of ka* is plotted. Since the largest value of the exact solution is not at the caustic, these errors are slightly larger than those of Table I. As ka becomes large, the error goes to zero. As ka goes to zero, μ_2 becomes singular. At small ka the exact solution approaches unity, and hence the relative error, $(\mu_2 - \mu_1)/\mu_1$, approaches the difference.

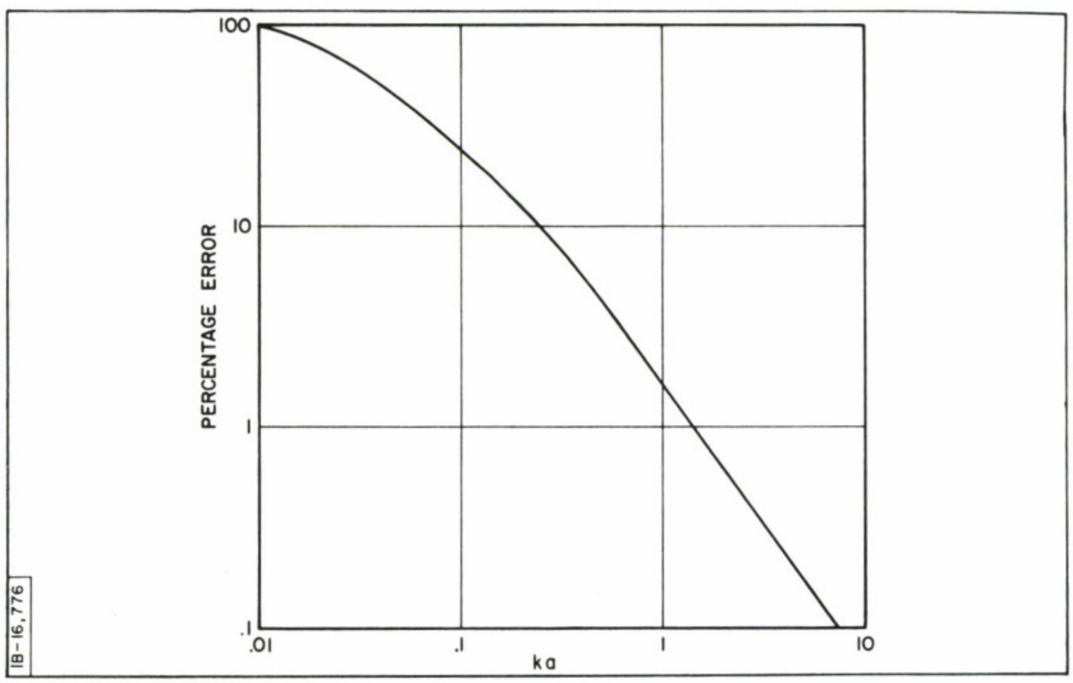

Figure 5. Plot of Error on the Caustic as a Function of ka

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* This curve was suggested by D. Ludwig, Courant Institute, New York, N.Y.

APPENDIX I

THE CIRCULAR CAUSTIC

Herein, aspects of the application of various representations of geometric optics fields near a caustic shall be considered. The standard asymptotic expansion and a representation due to R. M Lewis* shall also be considered. To be specific, the two-dimensional field that produces a circular caustic was chosen.

The wave front that produces a circular caustic is a spiral determined by a circle and constant length string. In Figure 6, we have the circular caustic of radius a. The points 1' and 2' lie on the spiral wave front. They are determined by the condition

$$\alpha_i + \rho_i = L , \tag{27}$$

where L is a constant and ρ_i is the distance on the wave front to the caustic; i.e., ρ_i is the magnitude of the line segment 11' .

The field at any point off the caustic is determined by the two rays that pass through the point. In Figure 2, the two indicated rays pass through the point P. The distance from the point i on the wave front to P is designated by τ_i . The distances from P to 1 and 2 are equal and are

$$S = \sqrt{r^2 - a^2} , \tag{28}$$

where $r \geq a$ is the radial coordinate of the point P. We note that

$$\tau_1 = \rho_1 - S , \tag{29}$$

$$\tau_2 = \rho_2 + S , \tag{30}$$

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* See page 1, first reference.

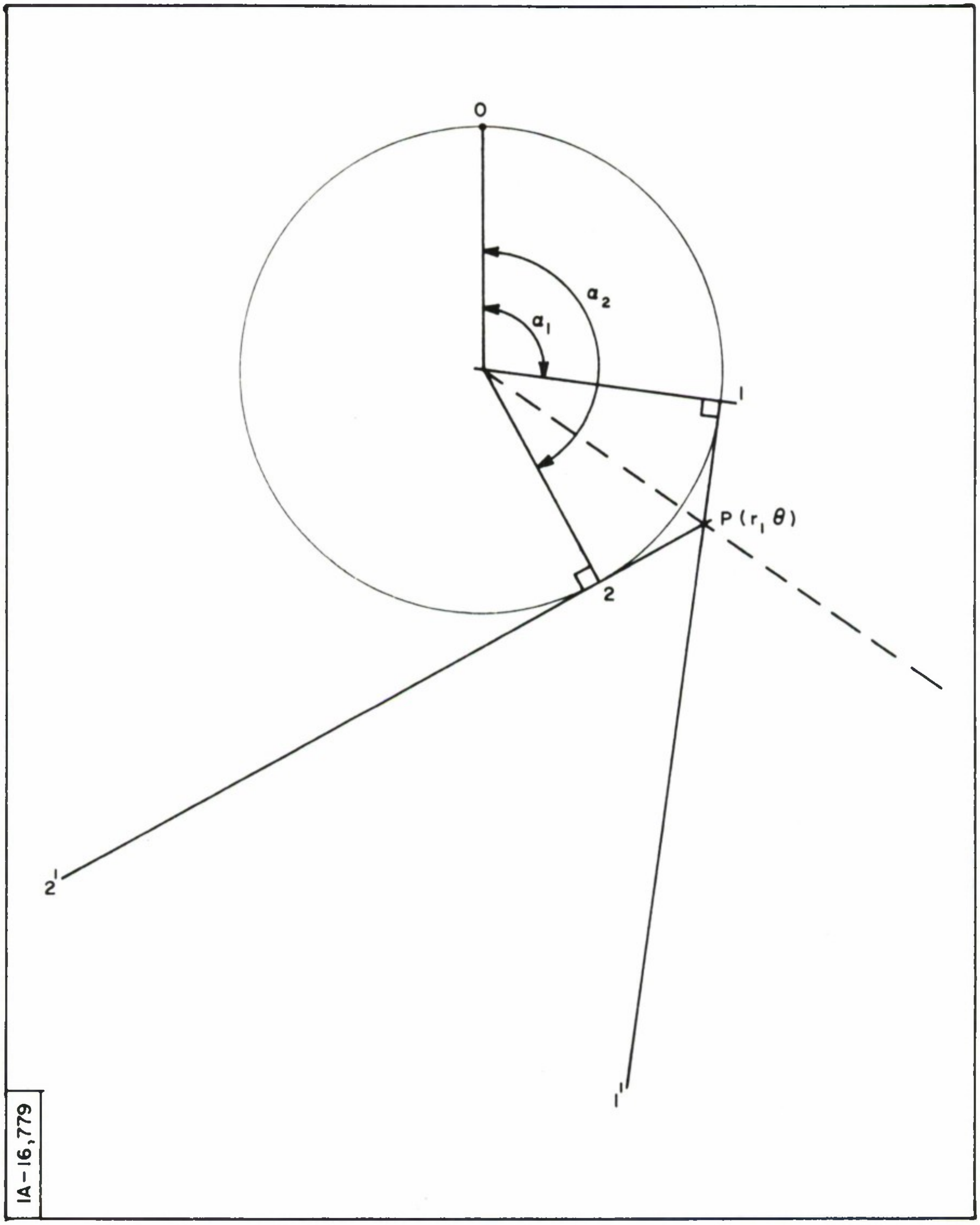

Figure 6. Circular Caustic

or that one ray through ρ has touched the caustic. The angular location of points 1 and 2 are found from

$$\phi = \theta \pm \cos^{-1} a/r \quad , \tag{31}$$

where θ is the angular coordinates of P. Using (30), (27), and (28) on (29), the following is obtained:

$$\tau_1 = L - a\left(\theta - \cos^{-1} a/r\right) - \sqrt{r^2 - a^2} \;; \tag{32}$$

$$\tau_2 = L - a\left(\theta - \cos^{-1} a/r\right) + \sqrt{r^2 - a^2} \;. \tag{33}$$

The standard asymptotic formula* for the field at ρ is

$$u = Z_1\left(1 - \tau_1/\rho_1\right)^{-1/2} e^{ik\left(s_1 + \tau_1\right)} + Z_2\left(1 - \tau_2/\rho_2\right)^{-1/2} e^{ik\left(s_2 + \tau_2\right)}, \tag{34}$$

where the subscript refers to the ray. This expression differs from that in Lewis' report* since this is a two-dimensional problem where ρ is used instead of the curvature κ. The term associated with the second ray involves taking the square root of a negative quantity. This indicates the phase shift of $\pi/2$ at the caustic. This formula is not valid at the caustic since it is singular at such points. Using the previous results for the circular caustic, relation (34) becomes

$$u = \frac{Z_1\left(r^2 - a^2\right)^{-1/4}}{\left(L - a\cos^{-1} a/r\right)^{1/2}} e^{ik\left(s_1 + L - a\theta + a\cos^{-1} a/r - \sqrt{r^2 - a^2}\right)}$$

$$+ \frac{Z_2\left(r^2 - a^2\right)^{-1/4}}{\left(1 - a\cos^{-1} a/r\right)^{1/2}} e^{ik\left(s_2 + L - a\theta - a\cos^{-1} a/r + \sqrt{r^2 - a^2} - \pi/2k\right)} \tag{35}$$

---

* See page 1, first reference, Equation (93).

For computation, the following assumptions are convenient,

$$Z_1 e^{iks_1} = Z_2 e^{iks_2} = k^{1/2}\left(L - a \cos^{-1} a/r\right)^{1/2} , \qquad (36)$$

$$\theta = 0 . \qquad (37)$$

Equation (37) requires that the field is constant along the wave front, and the results must be normalized so that this constant is unity. That $\theta = 0$ follows from a reasonable choice of coordinate system.

To simplify the result, L can now be chosen such that

$$e^{ikL} e^{-i\pi/4} = 1 . \qquad (38)$$

Then (35) reduces to

$$u = 2i \frac{e^{ikL} e^{+i\pi/4}}{k^{1/2}\left(r^2 - a^2\right)^{1/4}} \sin\left(ka \cos^{-1} a/r - k\sqrt{r^2 - a^2} + \pi/4\right), \qquad (39)$$

and Equation (39) is rewritten

$$u = \frac{2}{k^{1/2}\left(r^2 - a^2\right)^{1/4}} \sin\left(ka \cos^{-1} a/r - k\sqrt{r^2 - a^2} + \pi/4\right). \qquad (40)$$

Lewis* gives a uniform asymptotic expansion

$$u = \pi^{1/2} |\xi|^{1/2} e^{-i\pi/4} \left\{ Ai \; |-\xi^2| \left[\frac{Z_1 e^{ikb_1}}{\left|1 - \tau_1/\rho_1\right|^{1/2}} + \frac{Z_2 e^{ikb_2}}{\left|1 - \tau_2/\rho_2\right|^{1/2}} \right] \right.$$

$$\left. \frac{i\, Ai'\left(-\xi^2\right)}{\xi} \left[\frac{Z_1 e^{ikb_1}}{\left|1 - \tau_1/\rho_2\right|^{1/2}} - \frac{Z_2 e^{ikb_2}}{\left|1 - \tau_2/\rho_2\right|^{1/2}} \right] \right\} . \qquad (41)$$

---

* See page 1, first reference, Equations (68) and (69).

where

$$\zeta = (3\,k/4)^{1/3} \left(\tau_2 - \tau_1\right)^{1/3} \tag{42}$$

and

$$b_i = s_i + \frac{\tau_1 + \tau_2}{2} \tag{43}$$

Using Equations (32), (33), (36-38), and (41),

$$\mu = 2\pi^{1/2} \frac{\left|\left(3k/2\right)\left(\sqrt{r^2 - a^2} - a \cos^{-1} a/r\right)\right|^{1/6}}{k^{1/2}\left(r^2 - a^2\right)^{1/4}}$$

$$\mathrm{Ai}\left(-\left[\left(3k/2\right)\left|\sqrt{r^2 - a^2} - a \cos^{-1} a/r\right]^{2/3}\right) \tag{44}$$

Note the disappearance of the derivative of the Airy function because of the symmetry of the caustics. If the asymptotic expansion of the Airy function* is used in (44), Equation (40) is recovered.

The comparison of Equations (40) and (44) with Equations (22) and (23) shows agreement except for a numerical constant of $\sqrt{1/2\pi}$. Thus, a check is provided on the validity of the manipulations. This factor arises from the arbitrary choice of amplitude for the wave.

* See page 1, second reference, p. 33, Equation (30).

APPENDIX II

THE AIRY FUNCTION

The Airy function is defined (e. g. , in Jeffreys' work)* as

$$Ai(z) = \frac{1}{2\pi i} \int_{L_1} e^{tz - 1/3\, t^3}\, dt \ , \qquad (45)$$

where L_1 is the contour in the t-plane shown in Figure 7.

The transformation $t = is$ is introduced. Then for real z the transformed contour can be shifted to the real axis, and

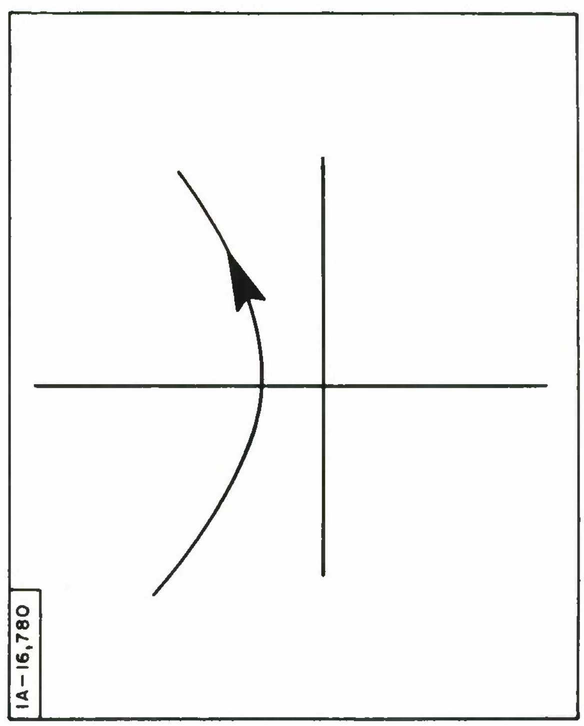

Figure 7. Contour Relevant to the Airy Function

* See page 1, second reference, Equation (1).

$$Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(xz+s/3)} ds \; ; \quad z \text{ real} \tag{46}$$

is obtained.

An alternative representation is

$$Ai(z) = \frac{1}{\pi} \int_{0}^{\infty} \cos(sz+s^3/3) \, ds \; ; \quad z \text{ real} \; . \tag{47}$$

APPENDIX III

THE UNIFORM ASYMPTOTIC FIELD AT THE CAUSTIC*

A quick examination of Equation (44) or, equivalently, (22) does not indicate that the uniform asymptotic expansion is finite at the caustic. In Lewis' report†, it was shown that this was generally true. Here, this shall be shown directly from Equation (22). The resultant formula will also give a simple expression near the caustic. Let

$$r = a\,(1 + \epsilon)\ , \tag{48}$$

where ϵ is a small positive quantity. Then

$$\sqrt{r^2 - a^2} = a\sqrt{2\epsilon + \epsilon^2} \sim a\sqrt{2\epsilon}\ , \tag{49}$$

and from relation (24)

$$\frac{\Psi - \sqrt{r^2 - a^2}}{a} = \cos^{-1}(a/r)\ . \tag{50}$$

Taking the cosine of both sides and expanding the power series to second order in ϵ and first order in Ψ ,

$$\Psi = \frac{2\sqrt{2}}{3}\ \frac{\epsilon^{3/2}}{a^{1/2}} \tag{51}$$

is obtained. For the field near the caustic, the following can now be obtained.

$$u = \left(\frac{2}{a}\right)^{1/3} \mathrm{Ai}\left[-\epsilon\left(\frac{2}{a}\right)^{1/3}\right]. \tag{52}$$

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*We wish to thank J. D. R. Kramer of The MITRE Corporation, Bedford, Mass. for this derivation.

† See page 1, first reference, Equation (6).

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13. ABSTRACT

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It is highly desirable that the abstract of classified reporta be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

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